# Solution of Two-Dimensional Electromagnetic Scattering Problem by FDTD with Optimal Step Size, Based on a Semi-Norm Analysis

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**Abstract.** To solve the electromagnetic scattering problem in two dimensions, the Finite Difference Time Domain (FDTD) method is used. The order of convergence of the FDTD algorithm, solving the two-dimensional Maxwell's curl equations, is estimated in two different computer implementations: with and without an obstacle in the numerical domain of the FDTD scheme. This constitutes an electromagnetic scattering problem where a lumped sinusoidal current source, as a source of electromagnetic radiation, is included inside the boundary. Confined within the boundary, a specific kind of Absorbing Boundary Condition (ABC) is chosen and the outside of the boundary is in form of a Perfect Electric Conducting (PEC) surface. Inserted in the computer implementation, a semi-norm has been applied to compare different step sizes in the FDTD scheme. First, the domain of the problem is chosen to be the free-space without any obstacles. In the second part of the computer implementations, a PEC surface is included as the obstacle. The numerical instability of the algorithms can be rather easily avoided with respect to the Courant stability condition, which is frequently used in applying the general FDTD algorithm.

**Keywords:** Electromagnetic scattering, Maxwell's equations, Finite Difference Time-Domain (FDTD) method, Courant stability condition, Absorbing boundary condition, point-wise convergence, semi-norm.

## INTRODUCTION

The numerical method of Finite Difference Time-Domain (FDTD) is widely used within the Computational Electromagnetics (CEM). Because of the simplicity of the method, FDTD is also applied to complicated structures like human body, antennas and wave-guides. Based on the Taylor's expansion method, finite difference equations are used to solve the Maxwell's coupled differential equations. In the well-known Yee formulation [1], the whole computational domain is divided, or discretized, according to so called leap-frog algorithm in which the volume elements are built as cubical cells. As a consequence, the area elements in the two-dimensional Yee formulation are built as rectangle/quadratic cells. To solve the differential form of Maxwell's equations by FDTD, the electric field variable $\mathbf{E}$ and the magnetic field variable $\mathbf{H}$ are computed at all locations in the discretized domain and at every time point. These finite difference expressions result in the FDTD leap-frog equations [2]. Choosing an initial time-step size, $\Delta t$, according to a defined stability condition, the FDTD equations are then solved by the following procedure [3]:

1. Calculating the electric field components $\mathbf{E}$ for the entire domain.
2. Advancing time by halved $\Delta t$.
3. Calculating the magnetic field components for the entire domain based on the electric field components calculated in 1.
4. Advancing time once again by halved $\Delta t$ and continuing to 1.

As a result, the field variables, $\mathbf{E}$ and $\mathbf{H}$, will be determined at all locations in the discretized domain and at every time point. Associated with the field variables $\mathbf{E}$ and $\mathbf{H}$, currents and voltages will finally be determined by post-processing.

In this paper, the authors give a convergence analysis method for a two-dimensional system excited by an incident electric field. The incident field, generated by an antenna, will be received by another antenna in another position within the two-dimensional system. Adding these two positions, i.e. the source point and the observation point, constitutes a fundamental stage in the numerical algorithm. An Absorbing Boundary Condition (ABC), confined in the boundary should also be added. By the ABC, a perfect absorption for traveling waves with any angle of incidence will be provided. To terminate the ABC space in practical applications, a perfect electric conductor (PEC) is used. For an acceptable cell size, cell dimensions in the algorithm should not be longer than one-tenth of the wave-length in question. However, the stability of the FDTD algorithm is guaranteed by considering the Courant stability condition [4]. To guarantee the convergence of the two-dimensional FDTD (2D-FDTD) algorithm, the authors give an analysis based on a point-wise convergence. In such context, a defined discrete norm is acquired by the computer implementation of the algorithm. In this process, the electric field at the observation point is computed for different spatial step sizes. As a result, the order of convergence, depending on the spatial step size, is determined by the computer implementation. It will be shown that both stability and convergence are guaranteed in the numerical algorithm. The 2D-FDTD algorithm is convergent also for the cases where a PEC surface, as an obstacle, is present in the two-dimensional domain. Computer simulations, based on the theoretical analysis, confirm stability and convergence of the numerical algorithm.

## THEORY

In constructing the electrostatic model, the electric field intensity vector **E** and the electric flux density vector, **D**, are respectively defined. The fundamental governing differential equations are [5]

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{D} &= \rho_v \end{aligned} \tag{1}$$

where $\rho_v$ is the volume charge density. By introducing $\varepsilon = \varepsilon_r \varepsilon_0$ as the the electric permittivity where $\varepsilon_r$ is relative permittivity, and $\varepsilon_0 = 8.854 \times 10^{-12} F/m$ as the permittivity of free space for a linear and isotropic media, **E** and **D** are related by relation

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{2}$$

The fundamental governing equations for magnetostatic model are

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \end{aligned} \tag{3}$$

where **B** and **H** are defined as the magnetic flux density vector and the magnetic field intensity vector, respectively. **B** and **H** are related as

$$\mathbf{H} = \frac{1}{\mu_0 \mu_r} \mathbf{B} \tag{4}$$

where $\mu_0 \mu_r = \mu$ is defined as magnetic permeability of the medium which is measured in $H/m.$; $\mu_0 = 4\pi \times 10^{-7}\ H/m$ is called permeability of free space and $\mu_r$ is a (material-dependent) number. The medium in question is assumed to be linear and isotropic. Eqns. (1) and (3) are known as Maxwell's equations and form the foundation of electromagnetic theory. As it is seen in the above relations, **E** and **D** in the electrostatic model are not related to **B** and **H** in the magnetostatic model. The coexistence of static electric fields and magnetic electric fields in a conducting medium causes an electromagnetostatic field and a time-varying magnetic field gives rise to an electric field. These are verified by numerous experiments. Static models are not suitable for explaining time-varying electromagnetic phenomenon. Under time-varying conditions it is necessary to construct an electromagnetic model in which the electric field vectors **E** and **D** are related to the magnetic field vectors **B** and **H**. In such situations, the equivalent equations are constructed as [6]

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{6}$$

$$\nabla \cdot \mathbf{D} = \rho_v \tag{7}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{8}$$

where $\mathbf{J} = \sigma \mathbf{E}$ is current density ($A/m^2$); $\sigma$ is called electric conductivity and is measured in Siemens (S) .

## The Finite Difference Time-Domain method

To reduce the three-dimensional Maxwell's equations into special forms with faster solution, a two-dimensional analysis is widely used. In this manner, the two-dimensional domain will be described in a mathematical sense though any radiating electromagnetic field takes place in the three-dimensional physical space. By assuming that there is no variation in the electromagnetic field in the $z$-direction, Maxwell's equation in Eqn. (6) can be expanded as [5]

$$-\mu\frac{\partial H_x}{\partial t}=\frac{\partial E_z}{\partial y}, \tag{9}$$

$$\mu\frac{\partial H_y}{\partial t}=\frac{\partial E_z}{\partial x}, \tag{10}$$

$$\frac{\partial E_z}{\partial t}=\frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}-\sigma E_z\right) \tag{11}$$

where the current density $\mathbf{J}$ is expressed as $\sigma E_z$ [5]. Eqns. (9)-(11) are referred to as Transverse Magnetic (TM) fields, or TM polarization fields, implying that the electromagnetic fields lie only in the $xy$-plane. Different electromagnetic field directions for the TM polarization are illustrated in Fig. 1, applying 2D-FDTD discretization. Consider a uniformly rectangular grid, $\Omega=[0,a]\times[0,b]$ where each grid cell has dimensions $\Delta x$ and $\Delta y$ along each Cartesian axis. By defining $\Delta t$ as the time-step, in the domain $(t,x,y)\in[0,T]\times\Omega$ for $T>0$, the following notation will be used in the 2D-FDTD formulation [7]

$$x_i=i\Delta x,\ x_{i+\frac{1}{2}}=x_i+\frac{1}{2}\Delta x,\ i=0,1,2,...,I-1,\ x_I=a, \tag{12}$$

$$y_j=j\Delta y,\ y_{j+\frac{1}{2}}=y_j+\frac{1}{2}\Delta y,\ j=0,1,2,...,J-1,\ y_J=b, \tag{13}$$

$$t^n=n\Delta t,\ t^{n+\frac{1}{2}}=t^n+\frac{1}{2}\Delta t,\ n=0,1,2,...,N-1,\ N\Delta t=T. \tag{14}$$

where $I$, $J$ and $N$ are positive integers so that the electric field at the spatial location $(i\Delta x, j\Delta y)$ and the time-step $t=n\Delta t$ is then denoted by

$$\mathbf{E}(i\Delta x,j\Delta y,n\Delta t)=\mathbf{E}^n_{i,j} \tag{15}$$

By the 2D-FDTD applied in this work, the staggered field quantities are based on a "leap-frog" time-update strategy where the magnetic and electric fields are computed in an altering manner. The method of Central Differences are used in FDTD to approximate the first-order derivatives in which the general derivative is written as [8]

$$\frac{\partial f(x)}{\partial x}=\frac{f\left(x+\frac{\Delta x}{2}\right)-f\left(x-\frac{\Delta x}{2}\right)}{\Delta x}+O(\Delta x^2) \tag{16}$$

or

$$\frac{\partial f(x)}{\partial x}\approx\frac{f\left(x+\frac{\Delta x}{2}\right)-f\left(x-\frac{\Delta x}{2}\right)}{\Delta x}. \tag{17}$$

By using the FDTD method, the TM polarization formulation in Eqns. (9)-(11), will then be substituted by the following difference equations [7]

$$H^{n+\frac{1}{2}}_{x_{i,j+\frac{1}{2}}}=H^{n-\frac{1}{2}}_{x_{i,j+\frac{1}{2}}}-\frac{\Delta t}{h\mu_0}\left[E^n_{z_{i,j+1}}-E^n_{z_{i,j}}\right], \tag{18}$$

$$H^{n+\frac{1}{2}}_{y_{i+\frac{1}{2},j}}=H^{n-\frac{1}{2}}_{x_{i+\frac{1}{2},j}}+\frac{\Delta t}{h\mu_0}\left[E^n_{z_{i+1,j}}-E^n_{z_{i,j}}\right], \tag{19}$$

$$E^{n+1}_{z_{i,j}}=E^n_{z_{i,j}}+\frac{\Delta t}{h\varepsilon_0}\left[H^{n+\frac{1}{2}}_{y_{i+\frac{1}{2},j}}-H^{n+\frac{1}{2}}_{y_{i-\frac{1}{2},j}}-H^{n+\frac{1}{2}}_{x_{i,j+\frac{1}{2}}}+H^{n+\frac{1}{2}}_{x_{i,j-\frac{1}{2}}}\right]-\frac{\Delta t}{h\varepsilon_0}J^{n+\frac{1}{2}}_{z_{i,j}} \tag{20}$$

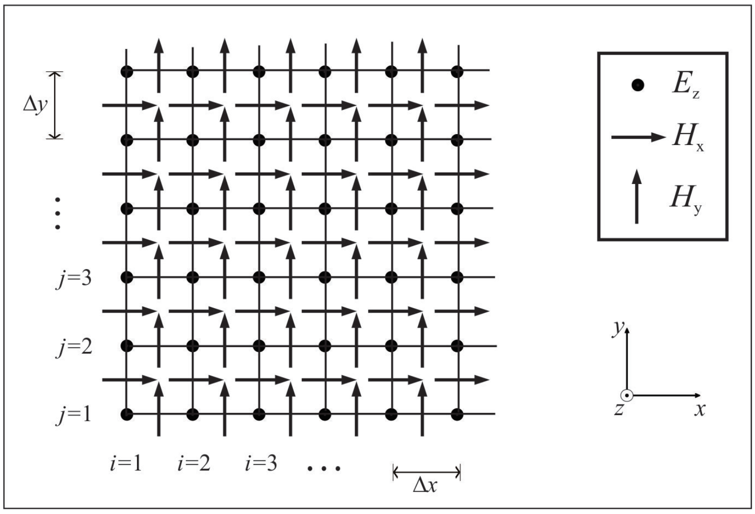


**FIGURE 1.** 2D-FDTD mesh for the transverse magnetic polarization.

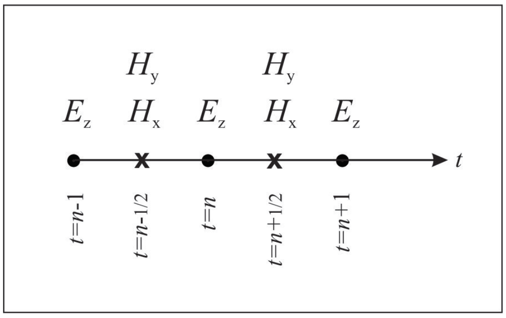


**FIGURE 2.** Leap-frog scheme used in 2D-FDTD for the transverse magnetic polarization.

where $h = \Delta x = \Delta y$ and $J_z$ can be a source due to an incident plane wave. The recursive update equations in Eqns. (18-20) are referred to as leap-frog time-update strategy in which $E_{z_{i,j}}^{n+1}$ relies on $H_{y_{i+\frac{1}{2},j}}^{n+\frac{1}{2}}$ and $H_{y_{i-\frac{1}{2},j}}^{n+\frac{1}{2}}$ along $y$-axis and $H_{x_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}}$ and $H_{x_{i,j-\frac{1}{2}}}^{n+\frac{1}{2}}$ along $x$-axis; these are spatially centered about $E_{z_{i,j}}^{n+1}$, see Fig. 1 and Fig. 2. On the other hand, the magnetic field quantities in both $x$-, and $y$-axis are updated at $n+\frac{1}{2}$, i.e., at the previous time-step for all space samples. The electric field quantities are then updated at time step $n+1$ for all space samples, and so on. To simulate unbounded problems by the FDTD method, an Absorbing Boundary Condition (ABC) should be designed. In [9] a so-called second-order accurate ABC is proposed to resolve the unboundedness of the problem. Considering the notations in (12-14) and Fig. 3, the left,- right,- and top absorbing boundaries in form of so-called MUR1, a first-order ABC, will then be written as [10]

$$E_{z_{0,j}}^{n+1} = E_{z_{1,j}}^{n} - C_{ABC}\left[E_{z_{1,j}}^{n+1} - E_{z_{0,j}}^{n}\right], \tag{21}$$

$$E_{z_{I,j}}^{n+1} = E_{z_{I-1,j}}^{n} - C_{ABC}\left[E_{z_{I-1,j}}^{n+1} - E_{z_{I,j}}^{n}\right], \tag{22}$$

$$E_{z_{i,J}}^{n+1} = E_{z_{i,J-1}}^{n} - C_{ABC}\left[E_{z_{i,J-1}}^{n+1} - E_{z_{i,J}}^{n}\right] \tag{23}$$

where $i = 0,1,2,...,I-1$, $j = 0,1,2,...,J-1$, and $C_{ABC} = (c_0\Delta t - h)/(c_0\Delta t + h)$ in which $c_0$ is light velocity in vacuum $(m/s)$ and $h = \Delta x = \Delta y$. To terminate the ABC space on the left, right and top, a perfect electric conductor (PEC) is used, see Fig. 3. A more complete first-order ABC, as well as the second-order are described in [10]. To simulate wave propagation by the 2D-FDTD algorithm, a line current, positioned in a point within the two-dimensional domain, is used. This time-dependent line current which can be a dipole,- or monopole antenna constitutes the source point in this formulation whose electric current can be chosen as

$$\mathbf{J}(t) = 1 - \cos(\frac{\pi t}{T}) \tag{24}$$

where $t$ is time in seconds and $T$ is the ramp time. In CEM simulation, a source can be designed in such a way that it acts during a very short time to excite the electromagnetic domain.

## Stability and convergence

To find the maximum stable time-step, a combination of theory and numerical experimentation is often the only way to proceed. To approximate the solution of a partial differential equation (PDE), one should analyze the problem by three important concepts which are consistency, convergence, and stability. By definition, a finite difference (FD) scheme is convergent if the point-wise error tends to zero as both the spatial and temporal step-sizes, for example $\Delta x$ and $\Delta t$, tend to zero. However, it may be cumbersome to prove convergence for many numerical schemes. In fact, the order of convergence gives a good indication to how well a solution of an FD scheme approximates the exact solution of a differential equation. An FD scheme is consistent if the truncation error tends to zero as the spatial and temporal step-sizes tend to zero. Truncation error gives therefore a rate to how well the exact solution of a differential equation satisfies the FD scheme. In addition, truncation error is an accessible rate to determine the accuracy of an FD scheme whose formal order of accuracy is defined by the order of its truncation error. Similar to the point-wise error, the truncation error is an accessible rate to the accuracy for an FD scheme. The point-wise error in an FD scheme shall grow unboundedly with time if the FD scheme is not stable. According to Lax Equivalence Theorem, a consistent FD scheme, approximating the exact solution, is convergent if and only if the FD scheme is stable [11]. The three-dimensional FDTD algorithm is stable according to Courant-Fredrichs-Lewy-, or CFL stability condition in which the time-step is bounded by the limit [12]

$$\Delta t \leq \frac{1}{c_0} \frac{1}{\sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}}} \tag{25}$$

However, in the 2D-FDTD algorithm in this work, the CFL stability condition in (25) is reduced into

$$\Delta t \leq \frac{h}{c_0} \frac{1}{\sqrt{2}} \tag{26}$$

where $h = \Delta x = \Delta y$. By choosing $\Delta t$, $\Delta x$, and $\Delta y$ based on the CFL stability condition, the 2D-FDTD algorithm will be convergent, according to the Lax Equivalence Theorem [13]. In problems applying central-finite differences, one does not know the order of convergence even though the standard central-difference algorithm converges with an error of order $h^2$; $h$ is the cell size. The order of convergence will be decreased due to singular behavior of the solution in problem with geometries possessing sharp edges of metallic conductors and dielectrics [14]. The next step will then be to determine the order of convergence which is introduced by an error analysis and defining a semi-norm.

## A semi-norm analysis and the order of convergence

The determine the order of convergence in the 2D-FDTD algorithm, an error analysis is used. A good way to visualize errors when they are expected to behave like some power of the spatial discretization, is to write the error $\varepsilon$ as [11]

$$\varepsilon(h) \approx C\,(h)^p \tag{27}$$

where $h$ is the cell size and $C$ is a constant independent of $h$. Then by taking logarithm of both sides of (27), namely

$$\log|\varepsilon\,(h)| \approx \log|C| + p\log|h| \tag{28}$$

the error $\varepsilon$, on a log-log scale, behaves linearly with a slope which is equal to the order of convergence $p$. The next step in the process of determining the order of convergence is to define a semi-norm.

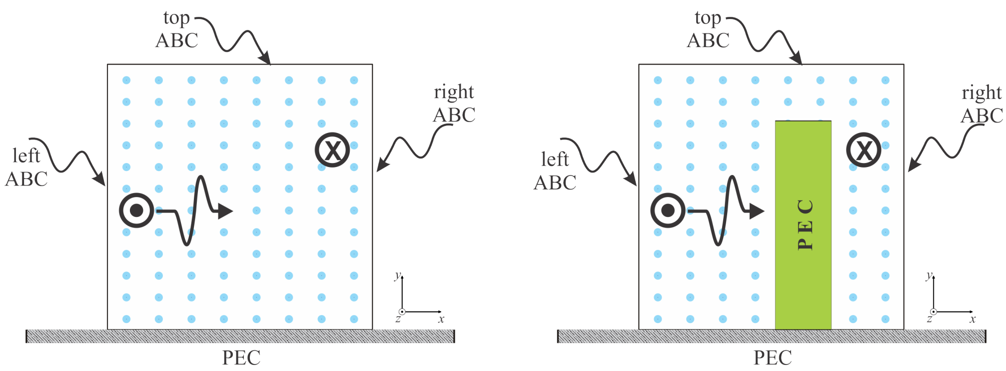


**FIGURE 3.** Absorbing boundary condition in form of MUR1 for the 2D-FDTD in two cases: without (left), or with (right) a PEC obstacle.

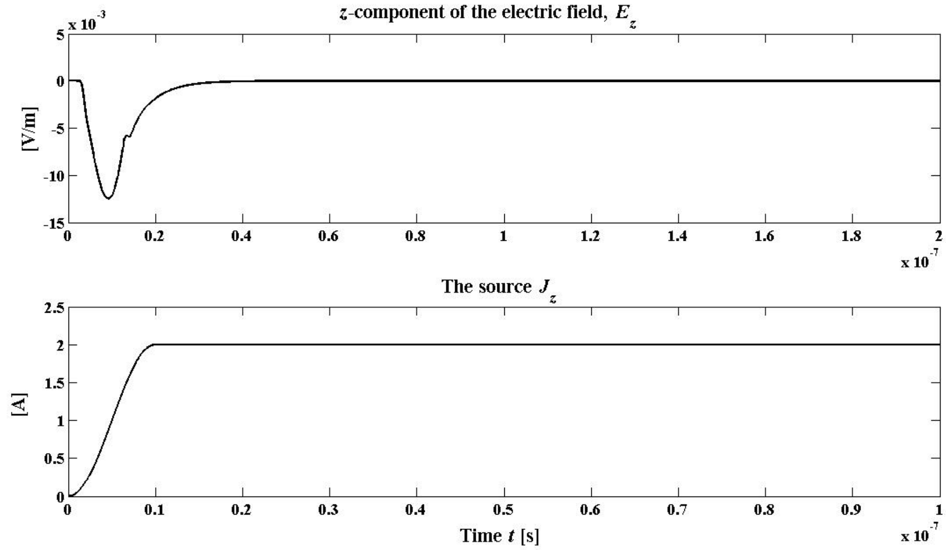


**FIGURE 4.** Typical computer simulation of the electric field due to an electric current source.

Let $X$ be a vector space. Then, for every $u, v \in X$, and every $c \in \mathbb{R}$, a semi-norm $|||.||| : X \to \mathbb{R}_+$ satisfies

$$\begin{aligned} N_1 &: |||u||| \geq 0 \\ N_2 &: |||cu||| = |c| \cdot |||u||| \\ N_3 &: |||u+v||| \leq |||u||| + |||v||| \end{aligned}$$

Let also that $P$ be an observation point. By defining the semi-norm $|||u||| = |u(P)|$ in this section, the goal is to show that the error $\varepsilon_h = |||u_h - u_{h/2}|||$ is of order 2, where $u_h$ is the numerical 2D-FDTD solution to the electric field $\mathbf{E}$ in the observation point $P$; $h = \Delta x = \Delta y$ is the spatial cell size in meter. Having (27) and (28) in mind, it will be shown that the error will be reduced by reducing the step size. Assume that

$$|||u_h - u_{\frac{h}{2}}||| \leq C \cdot h^k \tag{29}$$

where $C$ is some constant (independent on $k$), and the integer $k$ is to be determined by the above semi-norm; $u_{\frac{h}{2}}$ is the solution of the problem by a halved-cell size $\frac{h}{2}$. The discretized semi-norm will then be used to investigate how the order of convergence in the 2D-FDTD algorithm changes by halving the spatial step size $h$. The procedure for

**TABLE 1.** Discrete norm values for different spatial step-sizes $h = \Delta x = \Delta y$

| | $h$ | $\ln(h)$ | $\ln\left(\|\|\|u_h - u_{\frac{h}{2}}\|\|\|\right)$ |
|---|---|---|---|
| 1 | $\frac{1}{30}$ | -3.4012 | -3.2597 |
| 2 | $\frac{1}{60}$ | -4.0943 | -4.6670 |
| 3 | $\frac{1}{120}$ | -4.7875 | -6.0616 |
| 4 | $\frac{1}{240}$ | -5.4806 | -7.4570 |

investigating a point-wise convergence in the point $P$ is started by taking logarithm of both sides of Eqn. (29), that is

$$\begin{aligned} \ln\left(|||u_h - u_{\frac{h}{2}}|||\right) &\leq \ln\left(C \cdot h^k\right) \\ &= \ln(C) + \ln\left(h^k\right) \\ &= k\ln(h) + \ln(C). \end{aligned} \tag{30}$$

Eqn. (30) can be interpreted as a linear equation in the general form of $\eta = k\xi + m$, where $m = \ln(C)$, $\xi = \ln(h)$, and $\eta = \ln\left(|||u_h - u_{\frac{h}{2}}|||\right)$. The problem of determining the order of convergence is now converted to determine the slope $k$ in the linear equation of $\eta = k\xi + m$. The left hand-side of Eqn. (30) will then be found by running the computer program for different step sizes $h$. That is, at each step in the algorithm, the spatial step size $h$ (i.e. $\xi$-values) is halved.

To determine the order of convergence $k$ in the 2D-FDTD algorithm, data from TABLE 1 is used. Values for $\xi$ and $\eta$ can, in vector form, be written as

$$\xi = \ln(h) = [-3.4012 \; -4.0943 \; -4.7875 \; -5.4806] \tag{31}$$

$$\eta = \ln\left(|||u_h - u_{\frac{h}{2}}|||\right) = [-3.2597 \; -4.6670 \; -6.0616 \; -7.4570] \tag{32}$$

In this process, the time-step $\Delta t$ is chosen to have a constant value of $4ns$. To find order of the spatial acuracy $k$, in a least-squares sense, the two vectors in (31) and (32) should be fitted. As a result, the polynomial coefficients for the linear equation of $\eta = k\xi + m$, i.e. $k$ and $m$ will be determined as $k = 2.0178$ and $m = 3.5997$, respectively. This verifies that the 2D-FDTD algorithm is point-wise convergent by the order of 2 in the spatial discretization.

## CONCLUSIONS

To determine the order of convergence of the two-dimensional FDTD algorithm, a point-wise convergence analysis has been given in this work. In such context, a discrete norm has been defined which is acquired by the computer implementation of the algorithm. In this process, a discrete semi-norm, associated with the electric field at an observation point, is computed for different spatial step sizes. As a result, the order of convergence, depending on the spatial step size, has been calculated by a computer implementation. The spatial step sizes were chosen in a way that the two-dimensional FDTD algorithm was stable according to the Courant-Fredrichs-Lewy (CFL) stability condition.

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